Likelihood function for GPS movement data, normal inter-measurement times

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Abstract

Detecting movement in a sample of GPS measurements involves comparing changes in position with a measurements from a stationary device. I first show a sample of some data with discussion. I then model the the distribution function of the position changes with the objective of identifying moving and stationary states. I give the likelihood function derived from the statistical models. This first model used gamma distributions for times between measurements. This model uses truncated normal distributions. In previous demonstrations with the gamma distributions, it turned out that coefficients of variation were very low making the truncated normal model a practical alternative.

$\tau_i, i = 0, \ldots, n$ List of time stamps in a GPS track.

$\boldsymbol{x}_i, i = 0, \ldots, n$ List of global position measurements corresponding to the time stamps τ_i.

$T_i = \tau_i - \tau_{i-1}, i = 1, \ldots, n$ Time interval between successive measurements.

$S_i = d\left(\boldsymbol{x}_i, \boldsymbol{x}_{i-1}\right), i = 1, \ldots, n$ Distance between successive measurements.

σ_ε^2 Device position measurement error variance, each dimension, m^2.

V_i Speed with which device is actually moving before observation i.

$\bar{v}$ Mean speed of device when moving.

σ_v^2 Variance of device speed when moving.

μ_0, σ_0^2 Truncated normal distribution mean and variance parameters for time interval between measurements when device is at rest.

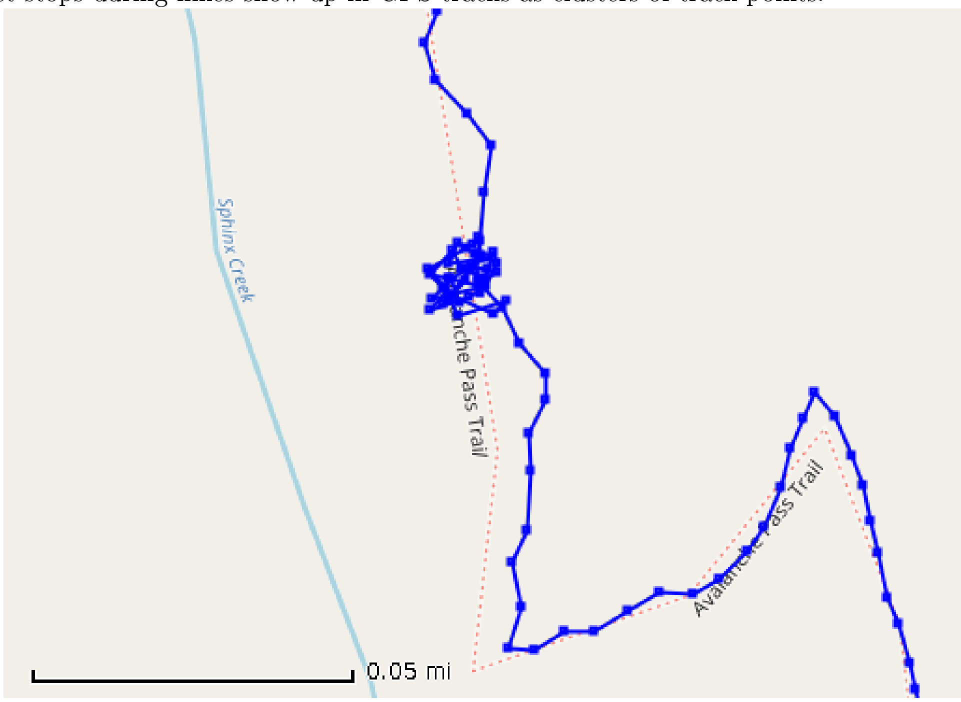
μ_1, σ_1^2 Truncated normal distribution mean and parameters for time interval between measurements when device is moving.

$f_{M=1}\left(s^2|T_i\bar{v}, T_i^2\sigma_v^2, \sigma_\varepsilon^2\right)$ The probability of density of $S_i$given that the navigation device is moving with mean speed $\bar{v}$ and variance σ_v^2.

$M_i, i = 1, \ldots, n$ Indicator of motion state of device in period before observation i. $M_i = 1$ indicates device was moving. If $M_i = 0$, it was stationary.

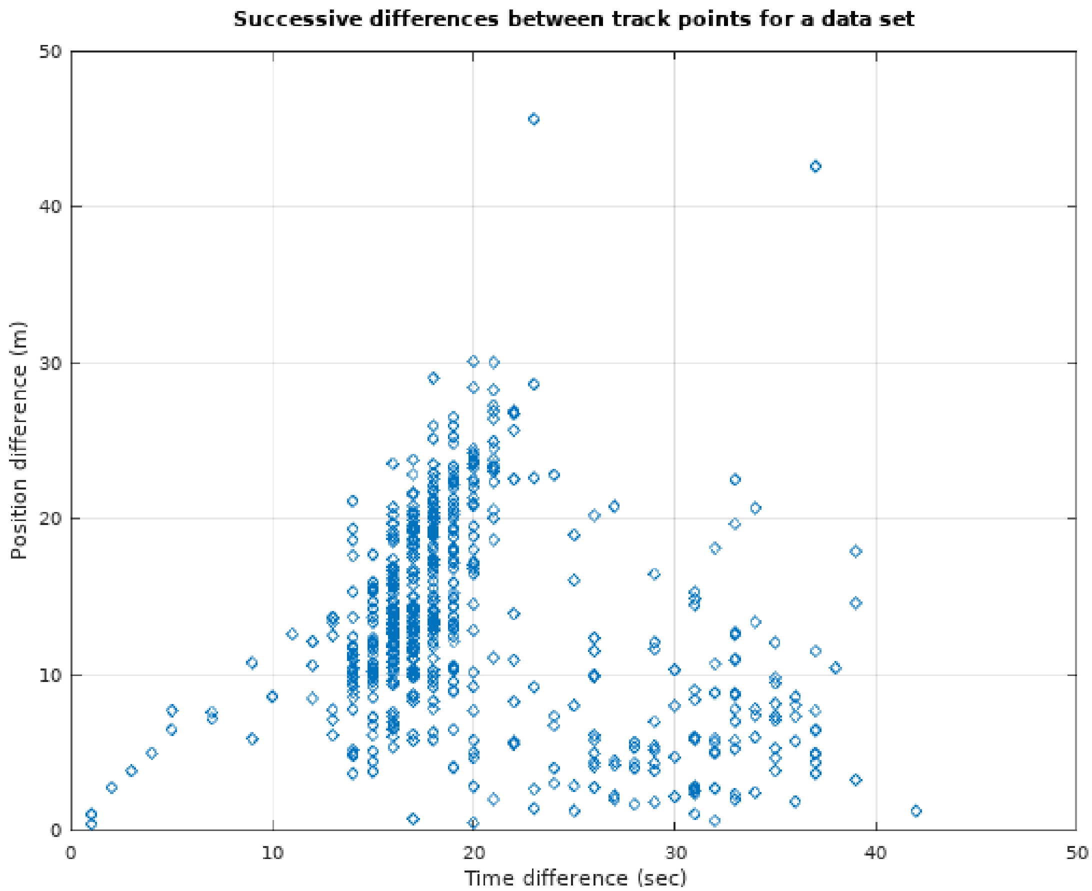
Movement modes and measurement

Rest stops during hikes show up in GPS tracks as clusters of track points.



Obviously, we were not zigzagging around during our rest stops. The GPS device was at rest, but it continued to take measurements with measurement errors of about ± 10 m. For a finer record of atheletic performance, it would be useful to identify intervals when the hiker rested. The problem is confounded by the presence of position measurement error. It is more difficult to identify rest intervals for foot travel than for, say, vehicular travel.

A GPS track consists of time-stamped positions $(\tau_i, \boldsymbol{x}_i)$, $i = 0, \ldots, n$. If we take successive differences on the times and positions, $T_i = \tau_i - \tau_{i-1}$, $S_i = d(\boldsymbol{x}_i, \boldsymbol{x}_{i-1})$, then we have series that can detect motion and even estimate speed. When the device is moving, S_i is the sum of its actual displacement over the time interval T_i and the difference of the position measurement errors. When the device is at rest, S_i contains only the difference in measurement errors and no real displacement. The following is a scatter diagram of one data set of successive differences from a backpack trip near Mount Brewer, CA. The device was a Garmin eTrex.



We see that the eTrex sample interval varied from 1 to 42 seconds, but was typically every 15 seconds. We can see two modes in the scatter diagram. One is centered at 18 seconds and 15 meters between measurements. The other mode is centered around 30 seconds and 5 meters between measurements. The latter mode appears to be associated with the device in the rest state, since the low position differences can be explained by measurement error only. It is notable that the rest mode has longer time intervals than the movement mode, but this can be explained by the behavior of the device algorithm ([ref1]).

Model of successive difference data

Each observation (T_i, S_i) is modeled as if independent of the others. The differences are not really independent, because they arose from successive differences. The assumption overstates the power of the data, but greatly simplifies our model. The assumption can be made correct by throwing out every other observation.

The device is either in a “stationary” (also called “rest”) state or a “moving” state.

If in the rest state, $S_i^2 \sim \sigma_\varepsilon^2 \times \chi^2 (\nu = 2)$, in other words, a chi-square distribution, two degrees of freedom, scaled by the position error measurement variance. This by virtue of S_i^2being the sum of of squares of two independent position measurement errors.

If in the moving state, the device moves at speed $V_i \sim N(\bar{v}, \sigma_v^2)$. In other words, if it is moving, the speed varies and is Gaussian.

In the moving state, S_i^2 has two components. One is aligned with the direction that the device moved and the other is measurement error orthogonal to the direction that the device moved. Thus, S_i^2 is the sum of the squares of $T_i V_i + Z_x \sigma_\varepsilon = T_i \bar{v} + Z_v T_i \sigma_v + Z_x \sigma_\varepsilon$ and $Z_y \sigma_\varepsilon$, where Z_v, Z_x, Z_y are independent standard normal random variables. $(T_i \bar{v} + Z_v T_i \sigma_v + Z_x \sigma_\varepsilon)^2$is the square of a $N(T_i \bar{v}, T_i^2 \sigma_v^2 + \sigma_\varepsilon^2)$, which makes it a non-central chi-square distribution $\left(\sigma_v^2 + \sigma_\varepsilon^2\right) \times \chi^2 \left(\nu = 1, \delta = \frac{T_i^2 \bar{v}^2}{\sigma_v^2 + \sigma_\varepsilon^2}\right)$. The second squared term, $(Z_y \sigma_\varepsilon)^2$, has a central chi-square distribution $\sigma_\varepsilon^2 \times \chi^2 (\nu = 1)$. Therefore, the distribution of $S_i^2$for the device in the moving state is a convolution of a non-central chi-square and central chi-square distributions with parameters as given in this paragraph. We give it the nomenclature

$$f_{M=1} \left(s^2 | T_i \bar{v}, T_i^2 \sigma_v^2, \sigma_\varepsilon^2\right),$$

and have implemented it by numerical integration in f_non_central_convolved within the notebook noncentralX2convolution.

The model treats time measurement errors as negligible. The variation of time between measurements, as seen in he scatter diagram is the result of pseudo-random behavior of the device algorithms. The time between measurements are assumed to be from two-parameter gamma distributions, one governing outputs when the device is at rest and the other for when the device is moving. For a device at rest, T_i has a $Normal \left(\mu_0, \sigma_0^2\right)$ distribution, and when the device is moving it has a $Normal \left(\mu_1, \sigma_1^2\right)$ distribution.

Likelihood function

Each term of the likelihood function has a form for a device at rest and for a device moving.

For the device at rest

If the device is at rest for observation i, then the likelihood function for observation i is

$$f_{M=0} \left(T_i, S_i^2\right) = \frac{1}{\sqrt{2\pi\sigma_0^2} \left(1 - \Phi\left(-\frac{\mu_0}{\sigma_0}\right)\right)} \exp\left(-\frac{(T_i - \mu_0)^2}{2\sigma_0^2}\right) \frac{1}{2\sigma_\varepsilon^2} \exp\left(-\frac{S_i^2}{2\sigma_\varepsilon^2}\right),$$

and the log likelihood function is

$$L_{M=0}\left(T_i,S_i^2\right)=-\log\left(1-\Phi\left(-\frac{\mu_0}{\sigma_0}\right)\right)-\frac{1}{2}\log\left(2\pi\sigma_0^2\right)-\frac{\left(T_i-\mu_0\right)^2}{2\sigma_0^2}-\log\left(2\sigma_\varepsilon^2\right)-\frac{S_i^2}{2\sigma_\varepsilon^2}.$$

For the device moving

If the device is moving for observation i, then the likelihood function is

$$f_{M=1}\left(T_i,S_i^2\right)=\frac{1}{\sqrt{2\pi\sigma_1^2}\left(1-\Phi\left(-\frac{\mu_1}{\sigma_1}\right)\right)}\exp\left(-\frac{\left(T_i-\mu_1\right)^2}{2\sigma_1^2}\right)f_{M=1}\left(s^2|T_i\bar{v},T_i^2\sigma_v^2,\sigma_\varepsilon^2\right),$$

and the log likelihood function is

$$L_{M=1}\left(T_i,S_i^2\right)=-\log\left(1-\Phi\left(-\frac{\mu_1}{\sigma_1}\right)\right)-\frac{1}{2}\log\left(2\pi\sigma_1^2\right)-\frac{\left(T_i-\mu_1\right)^2}{2\sigma_1^2}+\log\left(f_{M=1}\left(s^2|T_i\bar{v},T_i^2\sigma_v^2,\sigma_\varepsilon^2\right)\right).$$

The combined likelihood function

Let $M_i, i=1,\ldots,n$ be the indicator of the motion state of the device. We can use M_i to describe the likelihood function for all of the motion data.

$$L\left(\left\{\left(T_i,S_i^2\right)\right\}_{i=1}^n|\left\{M_i\right\}_{i=1}^n,\bar{v},\sigma_v,\sigma_\varepsilon^2\right)$$

$$=\sum_{i=1}^n(1-M_i)\left(-\log\left(1-\Phi\left(-\frac{\mu_0}{\sigma_0}\right)\right)-\frac{1}{2}\log\left(2\pi\sigma_0^2\right)-\frac{\left(T_i-\mu_0\right)^2}{2\sigma_0^2}-\log\left(2\sigma_\varepsilon^2\right)-\frac{S_i^2}{2\sigma_\varepsilon^2}\right)$$

$$+M_i\left(-\log\left(1-\Phi\left(-\frac{\mu_1}{\sigma_1}\right)\right)-\frac{1}{2}\log\left(2\pi\sigma_1^2\right)-\frac{\left(T_i-\mu_1\right)^2}{2\sigma_1^2}+\log\left(f_{M=1}\left(s^2|T_i\bar{v},T_i^2\sigma_v^2,\sigma_\varepsilon^2\right)\right)\right) \tag{1}$$

Maximum Likelihood Estimation

In (1), I have delineated $\left\{M_i\right\}_{i=1}^n,\bar{v},\sigma_v,\sigma_\varepsilon^2$ as principal parameters to be estimated. Others, $\mu_0,\sigma_0^2,\mu_1,\sigma_1^2$, are nuisance parameters which also would be estimated. Given a particular solution for just the scalar parameters, say $\hat{\bar{v}},\hat{\sigma}_v,\hat{\sigma}_\varepsilon^2,\hat{\mu}_0,\hat{\sigma}_0^2,\hat{\mu}_1,\hat{\sigma}_1^2$, and holding those fixed, it is a trivial matter to maximize the likelihood by choosing M_i. Since each M_i affects exactly one term of the log likelihood, choose $\hat{M}_i$ to maximize that one term thus:

$$\hat{M}_i = \underset{M=0,1}{\arg\max}\ [-\log\left(1-\Phi\left(-\frac{\mu_0}{\sigma_0}\right)\right)-\frac{1}{2}\log\left(2\pi\sigma_0^2\right)-\frac{\left(T_i-\mu_0\right)^2}{2\sigma_0^2}-\log\left(2\sigma_\varepsilon^2\right)-\frac{S_i^2}{2\sigma_\varepsilon^2},$$
$$-\log\left(1-\Phi\left(-\frac{\mu_1}{\sigma_1}\right)\right)-\frac{1}{2}\log\left(2\pi\sigma_1^2\right)-\frac{\left(T_i-\mu_1\right)^2}{2\sigma_1^2}-\log\left(f_{M=1}\left(s^2|T_i\bar{v},T_i^2\sigma_v^2,\sigma_\varepsilon^2\right)\right)]$$

Inter-measurement time constraint

This model has a constraint on the left-tail probabilities of the distributions for the time between measurements.

$$P_{\mu_0,\sigma_0^2}\left(T_i < t\right) < P_{\mu_1,\sigma_1^2}\left(T_i < t\right). \quad (2)$$

In words, the distribution of inter-measurement times for the stationary device is to the left of the one for the moving device. (2) need only hold for the left-tail probabilities, for those values of t for which $P_{\mu_1,\sigma_1^2}\left(T_i < t\right) < 0.5$. For unrestricted normal distributions, (2) is equivalent to

$$\mu_0 \geq \mu_1 \text{and} \sigma_0^2 \leq \sigma_1^2. \quad (3)$$

(2) is not so true for truncated normal distributions, but the truncation probabilities $\Phi\left(-\frac{\mu_0}{\sigma_0}\right), \Phi\left(-\frac{\mu_1}{\sigma_1}\right)$ being small, we use (3) in place of (2).

References

[ref1] http://www.gpsinformation.org/dale/tracklog.htm