Note: Convolution of a non-central chi-square with a central chi-square distribution

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Abstract

I need an expression for the convolution of a non-central chi-square distribution with a central chi-square, where the two have different scale factors and different degrees of freedom. The expression will be used to implement a Python function of the result's density function for use in maximum likelihood estimation of GPS movement data.

ν_X, ν_Y Degrees of freedom of the two distributions.

σ_X^2, σ_Y^2 Scale factors of the two distributions.

δ_X Non-centrality parameters of the first distribution. The second distribution is an ordinary central chi-square.

X, Y Two input random variables to the problem.

f_X, f_Y Probability density functions of the two random variables.

Problem

X is a non-central chi-square distribution with non-centrality parameter δ_X. Y is an ordinary central chi-square distribution. The distributions are multiplied by different scale factors.

$$X \sim \sigma_X^2 \chi^2 \left(\nu_x, \delta_X\right)$$

$$Y \sim \sigma_Y^2 \chi^2 \left(\nu_Y\right).$$

The support of these distributions is the positive real line and

$$f_X\left(x\right) = \frac{1}{2\sigma_X^2}\left(\frac{x}{\delta_X \sigma_X^2}\right)^{\frac{\nu_x}{4}-\frac{1}{2}} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma_X^2}+\delta_X\right)\right) I_{\frac{\nu_x}{2}-1}\left(\sqrt{\frac{\delta_X x}{\sigma_X^2}}\right)$$

$$f_Y(y) = \frac{1}{\sigma_Y^2} \frac{1}{2^{\frac{\nu_Y}{2}} \Gamma\left(\frac{\nu_Y}{2}\right)} \left(\frac{y}{\sigma_Y^2}\right)^{\frac{\nu_Y}{2}-1} \exp\left(-\frac{y}{2\sigma_Y^2}\right).$$

where $I_n()$ is a modified Bessel function of the first kind, and $\Gamma()$is the gamma function.

Now suppose $Z = X + Y$. Z also has support on the positive real line. The problem is to calculate the density of Z at any point on the positive real line.

Convolution Integral

For any $z > 0$, the density of Z is

$$f_Z(z) = \frac{1}{2\sigma_X^2 \sigma_Y^2 2^{\frac{\nu_Y}{2}} \Gamma\left(\frac{\nu_Y}{2}\right)} \int_0^z \left(\frac{z-y}{\delta_X \sigma_X^2}\right)^{\frac{\nu_x}{4}-\frac{1}{2}} \left(\frac{y}{\sigma_Y^2}\right)^{\frac{\nu_Y}{2}-1} \exp\left(-\frac{1}{2}\left(\frac{z-y}{\sigma_X^2} + \delta_X + \frac{y}{\sigma_Y^2}\right)\right) I_{\frac{\nu_x}{2}-1}\left(\sqrt{\frac{\delta_X (z-y)}{\sigma_X^2}}\right) dy. \tag{1}$$

Solution by numerical integration

In my GPS applications, $\nu_Y = 1$. The variable is the squared measurement error from one dimension. This means that the integrand has a pole at $y = 0$. To make it easier for numerical methods to manage integration intervals near the left-hand limit, we can transform y to a semi-infinite domain where the integrand is finite. Apply the change of variables

$$y = \frac{1}{u}$$

$$dy = -\frac{1}{u^2}$$

$$y \in (0, z) \rightarrow u \in \left(\frac{1}{z}, +\infty\right)$$

$$f_Z(z) = \frac{1}{2\sigma_X^2 \sigma_Y^2 2^{\frac{\nu_Y}{2}} \Gamma\left(\frac{\nu_Y}{2}\right)} \int_{\frac{1}{z}}^{\infty} \left(\frac{z-\frac{1}{u}}{\delta_X \sigma_X^2}\right)^{\frac{\nu_x}{4}-\frac{1}{2}} \left(\frac{1}{u\sigma_Y^2}\right)^{\frac{\nu_Y}{2}-1} \exp\left(-\frac{1}{2}\left(\frac{z-\frac{1}{u}}{\sigma_X^2} + \delta_X + \frac{1}{u\sigma_Y^2}\right)\right) I_{\frac{\nu_x}{2}-1}\left(\sqrt{\frac{\delta_X\left(z-\frac{1}{u}\right)}{\sigma_X^2}}\right) \frac{du}{u^2}.$$

The scipy.integrate function quad provides a general purpose numerical integrator. It requires a function to evaluate the integrand.

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def f_non_central_joint(x, nuX, sigmaX2, deltaX, nuY, sigmaY2, z):
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f_non_central_joint has six parameters in addition to x, which are needed to evaluate the integrand,

$$\left(\nu_x, \sigma_X^2, \delta_X, \nu_Y, \sigma_Y^2, z\right).$$

The scipy.special function iv(v,z) calculates the modified Bessel function of the first kind with real-valued order.

The scipy.special function gamma(z) calculates the gamma function.

A particular application

In the particular GPS movement detection application, $\nu_X = \nu_Y = 1$, $\sigma_X^2 = T^2\sigma_v^2$, $\sigma_Y^2 = \sigma_\varepsilon^2$, $\delta_X = \frac{\bar{v}^2}{\sigma_v^2}$ and $z = S^2$. The convolution integral (1) becomes

$$f\left(s^2|T^2\bar{v}^2, T^2\sigma_v^2, \sigma_\varepsilon^2\right) = \frac{1}{2T^2\sigma_v^2\sigma_\varepsilon^2 2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)}\int_0^{S^2}\left(\frac{S^2-y}{\frac{\bar{v}^2}{\sigma_v^2}T^2\sigma_v^2}\right)^{-\frac{1}{4}}\left(\frac{y}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\frac{S^2-y}{T^2\sigma_v^2}+\frac{\bar{v}^2}{\sigma_v^2}+\frac{y}{\sigma_\varepsilon^2}\right)\right) I_{-\frac{1}{2}}\left(\sqrt{\frac{\frac{\bar{v}^2}{\sigma_v^2}(S^2-y)}{T^2\sigma_v^2}}\right)dy$$

$$= \frac{1}{2T^2\sigma_v^2\sigma_\varepsilon^2 2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)}\int_0^{S^2}\left(\frac{S^2-y}{T^2\bar{v}^2}\right)^{-\frac{1}{4}}\left(\frac{y}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\frac{S^2-y}{T^2\sigma_v^2}+\frac{\bar{v}^2}{\sigma_v^2}+\frac{y}{\sigma_\varepsilon^2}\right)\right) I_{-\frac{1}{2}}\left(\frac{\bar{v}}{T\sigma_v^2}\sqrt{S^2-y}\right)dy.$$

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